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A note on the solution of the Navier–Stokes equations for a spherically symmetric expansion into a very low pressure

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It is shown that, for a spherically symmetric expansion of a gas into a low pressure, the shock wave with area change region discussed earlier (Freeman & Kumar 1972) can be further divided into two parts. For the Navier–Stokes equation, these are a region in which the asymptotic zero-pressure behaviour predicted by Ladyzhenskii is achieved followed further downstream by a transition to subsonic-type flow. The distance of this final region downstream is of order (pressure)$^{-\frac{2}{3}}$ × (Reynolds number)$^{-\frac{1}{3}}$.

1. Introduction

In a recent paper the authors (Freeman & Kumar 1972, subsequently referred to as I) described the spherically symmetric expansion of a viscous heat-conducting gas into a region of low pressure for small Reynolds number according to the Navier–Stokes equations.

It was shown that, in terms of the inverse Reynolds number α based on sonic conditions (denoted by suffix $*$), the motion could be described by three regions: an inviscid region characterized by r/r_* of order one, an intermediate region where r/r_* was of order $\alpha^{-\mu}$ with $\mu = [2\gamma - 1 + 2\omega(\gamma-1)]^{-1}$ and γ the ratio of specific heats and a shock layer where r/r_* was of order α^{-1}. Satisfactory matching was achieved between these regions and it was shown that the full numerical solutions of the equations exhibited this structure. The flow in the final shock-layer region was assumed to terminate at infinity in a subsonic-type flow where the pressure was finite but small. Since according to the ideal-gas law the pressure p is related to the density ρ and temperature T by

$$p = \rho RT,$$

with R the gas constant, and the mass flow is conserved by the relation

$$M = \rho u r^2,$$

where M is the constant mass flow and u is the radial velocity, we may write

$$p = MRT/ur^2.$$

Introducing non-dimensional quantities $P = p/p_*$, $\theta = T/T_*$ and $w = u/u_*$, this becomes

$$P = \theta x^2/w, \tag{1.1}$$

where $x = r_*/r$ since $M = \rho_* u_* r_*^2$ and $p_* = \rho_* RT_*$. Thus the requirement that the pressure remains significant in the shock layer demands that P shall be $O(\alpha^2)$ if θ and w are of order one and x is of order α. Using $Y = x/\alpha$ as the variable in the shock layer we obtain

$$P/\alpha^2 = \theta Y^2/w. \tag{1.2}$$

The background pressure must be of order α^2 therefore. The shock-layer flow is essentially a region where area change and dissipation become equally important and hence our concern is essentially in obtaining the shock-wave structure including the area change. Since the asymptotic behaviour downstream of the shock wave is what one would, in inviscid terms, describe as subsonic, we require that $w/Y^2 \to W_0$, a constant, and $\theta \to \frac{1}{2}(\gamma+1)$ as $Y \to 0$. As the pressure level is inversely proportional to W_0, the zero-pressure limit is thus characterized by $W_0 \to \infty$.

It has been pointed out by Ladyzhenskii (1962) that the limiting behaviour at infinity of the Navier–Stokes equations for α finite is not the supersonic inviscid limit ($\alpha = 0$). Indeed it is possible to show that no solution with w finite exists as $Y \to 0$. Ladyzhenskii suggests that the correct asymptotic behaviour for zero pressure is given by a solution that has θ finite but velocity zero in such a way that $w/\sqrt{Y}$ remains finite. Since the approach of the velocity to zero is slower than the subsonic inviscid result this implies that the pressure is zero from (1.1). This might be expected to be the limiting behaviour therefore for the limit W_0 infinite. If this is the case then the nature of the non-uniformity associated with $W_0 \to \infty$, $Y \to 0$ is immediately evident since we require that

$$W_0 Y^2 = O(\sqrt{Y}),$$

or

$$Y = O(W_0^{-\frac{2}{3}});$$

and

$$w = O(W_0 Y^2) = O(W_0^{-\frac{1}{3}}).$$

Since θ approaches $\frac{1}{2}(\gamma+1)$ in a subsonic-type limit and is constant in the zero-pressure limit, it is to be expected that θ remains constant to first order in this region.

It remains to decide whether such a variation is consistent with the equations of motion.

2. Shock-layer structure

The Navier–Stokes equations in the shock layer may be written as

$$\theta^\omega\left(\frac{d^2w}{dY^2}-\frac{2w}{Y^2}\right)+\omega\theta^{\omega-1}\frac{d\theta}{dY}\left(\frac{dw}{dY}+\frac{w}{Y}\right)+\frac{dw}{dY} = -\frac{1}{\gamma}\left\{\frac{d}{dY}\left(\frac{\theta}{w}\right)+\frac{2}{Y}\frac{\theta}{w}\right\} \tag{2.1}$$

and

$$\theta+\frac{\gamma-1}{2}\left(1+\frac{2\theta^\omega}{Y}\right)w^2-\frac{\gamma+1}{2} = \theta^\omega\left\{\frac{3}{4\sigma}\frac{d\theta}{dY}+\frac{\gamma-1}{2}\frac{dw^2}{dY}\right\} \tag{2.2}$$

with boundary conditions

$$w/Y^2 \to W_0, \quad \theta \to \tfrac{1}{2}(\gamma+1) \quad \text{as} \quad Y \to 0,$$

where σ is the Prandtl number.

The matching as $Y \to \infty$ has already been discussed in I. Our concern is with the limit $W_0 \to \infty$ near $Y = 0$. Using the information deduced above, we write

$$w = W_0^{-\frac{1}{3}} W, \quad z = Y W_0^{\frac{2}{3}}, \quad \theta = \tfrac{1}{2}(\gamma+1) + \phi W_0^{-\frac{2}{3}}. \tag{2.3}$$

Substituting in (2.1) and (2.2) gives to first order as $W_0 \to \infty$

$$\left(\frac{\gamma+1}{2}\right)^{\omega}\left[\frac{d^2W}{dz^2} - \frac{2W}{z^2}\right] = \frac{\gamma+1}{2\gamma}\left[\frac{1}{W^2}\frac{dW}{dz} - \frac{2}{zW}\right] \tag{2.4}$$

and

$$(\gamma-1)\frac{W^2}{z} = \left[\frac{3}{4\sigma}\frac{d\phi}{dz} + \frac{\gamma-1}{2}\frac{dW^2}{dz}\right]. \tag{2.5}$$

The boundary conditions are given by

$$W/z^2 = 1 \quad \text{as} \quad z \to 0, \qquad W/z^{\frac{1}{3}} = \text{constant as } z \to \infty, \tag{2.6}$$

where the constant is given by the matching with the asymptotic expansion upstream. Writing (2.4) as

$$\frac{d^2W}{dz^2} - \frac{2W}{z^2} = \Gamma\left[\frac{1}{W^2}\frac{dW}{dz} - \frac{2}{zW}\right], \tag{2.7}$$

where $\Gamma = \gamma^{-1}[\frac{1}{2}(\gamma+1)]^{1-\omega}$, we obtain

$$W/z^{\frac{1}{3}} \to (\tfrac{2}{3}\Gamma)^{\frac{1}{3}} \quad \text{as} \quad z \to \infty.$$

Equation (2.7) may be written in a simpler form if the variable $U = W/z^2\Gamma^{\frac{1}{2}}$ is introduced. It then becomes

$$U'' + \frac{U'}{z^2}\left(4z - \frac{1}{z^2U^2}\right) = 0, \tag{2.8}$$

with $U = \Gamma^{-\frac{1}{2}}$ at $z = 0$; $Uz^{\frac{5}{3}} = (\frac{2}{3})^{\frac{1}{3}}$ at $z = \infty$.

This equation has been obtained by Bush & Rosen (1971) in their solution for the expansion into a vacuum in the case $\gamma = 1$. It can be integrated directly to give

$$z^4\,dU/dz + 1/U = A, \tag{2.9}$$

where A is a constant.

The solution which satisfies the given boundary conditions is

$$\frac{U}{\Gamma^{\frac{1}{2}}} + \frac{1}{\Gamma}\log(1 - \Gamma^{\frac{1}{2}}U) = -\frac{1}{3z^3}, \tag{2.10}$$

or

$$\frac{W}{z^2} + \log\left(1 - \frac{W}{z^2}\right) = -\frac{\Gamma}{3z^3}. \tag{2.11}$$

Rewritten in terms of unscaled variables this becomes

$$\frac{w}{W_0Y^2} + \log\left(1 - \frac{w}{W_0Y^2}\right) = -\frac{\Gamma}{3Y^3W_0^2}, \tag{2.12}$$

from which the two asymptotic behaviours are obtained in the limits as

$$\left.\begin{aligned} w &= W_0Y^2 \quad &\text{as} \quad z \to 0, \\ w &= Y^{\frac{1}{3}}(\tfrac{2}{3}\Gamma)^{\frac{1}{3}} \quad &\text{as} \quad z \to \infty. \end{aligned}\right\} \tag{2.13}$$

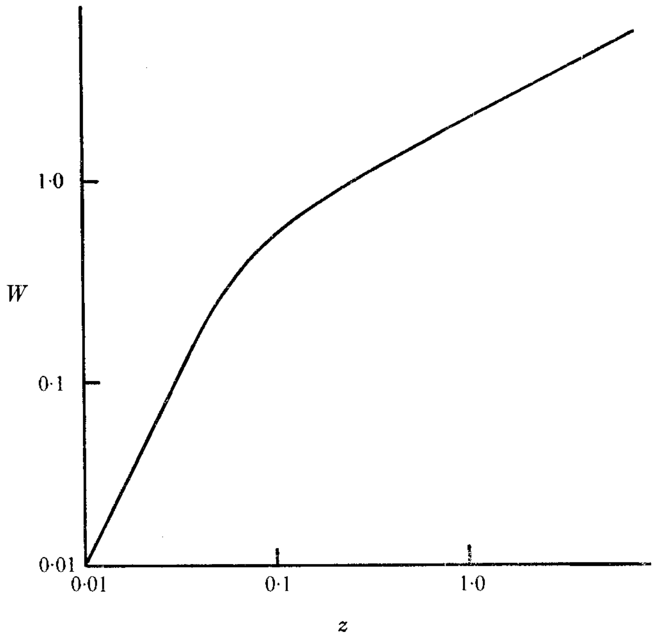

FIGURE 1. Velocity in final layer from equation (2.11) (logarithmic scales).

This solution thus matches the first terms of the two expansions. It is, of course, necessary that subsequent terms of the expansion are themselves uniformly valid in the appropriate limits.

W. B. Bush (private communication) has indicated that the higher order terms of the Ladyzhenskii expansion do not give a uniformly valid expansion. In particular, he shows that, relative to the first term, the term in the expansion of order $\alpha^{\frac{3}{2}}$ behaves like $Y^{-\frac{3}{2}}$ for $Y \to 0$. This implies that a breakdown will occur in a region where $Y = O(\alpha)$. Since the previous theory assumes that $Y = O(W_0^{-\frac{2}{3}})$, this would impose a limitation on the previous theory that $W_0 = o(\alpha^{-\frac{3}{2}})$. Or, in physical terms, a limitation on the ambient pressure, which is $O(\alpha^2/W_0)$, to greater than $O(\alpha^{\frac{7}{2}})$.

The behaviour of the velocity from (2.11) is shown in figure 1 for $\gamma = \frac{5}{3}$, $\omega = \frac{3}{4}$ and $\sigma = \frac{3}{4}$.

3. Numerical results

Results were given in I for values of W_0 of 1, 10 and 100 (for $\gamma = \frac{5}{3}$ and $\sigma = \omega = \frac{3}{4}$). These results, together with a further computation for $W_0 = 1000$, are shown in figure 2. These may be compared with the results shown in figure 1 in terms of scaled variables. Since the plot is logarithmic, the effect of varying W_0 is simply to displace the curves by a constant amount in each direction.

The nature of the solution in the shock layer itself in the limit of $W_0 \to \infty$ is more difficult to obtain since it is necessary to integrate the full equations between

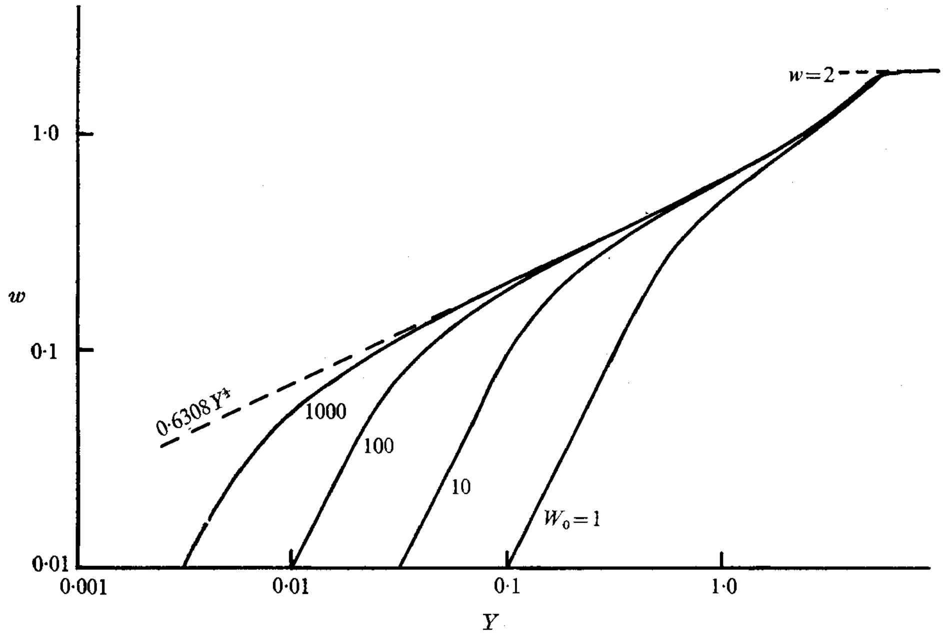

FIGURE 2. Velocity in shock layer and final layer from numerical solutions (logarithmic scales).

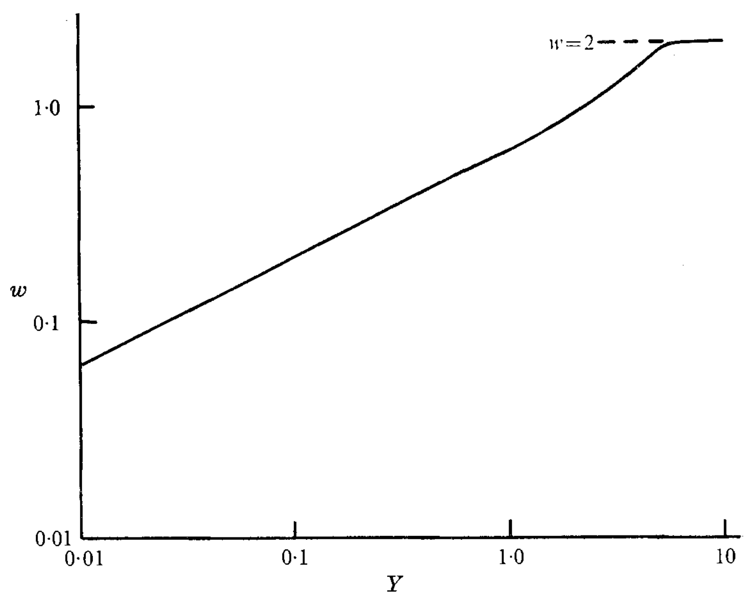

FIGURE 3. Velocity in shock layer: zero-pressure solution (logarithmic scales).

two singular points. The nature of this limiting behaviour can however be observed by looking at the variation of the numerical solutions with W_0 as in figure 3.

4. Conclusion

It has been shown how in the limit of very low pressure the structure of the flow field can be obtained for the Navier–Stokes equations. It is of interest to compare these results with the work of Brook & Hamel (1972), who attempt to construct such a solution for the Boltzmann equation. They introduce two length scales into the problem. These are R_∞, which represents the free path of a molecule emanating from the source in a uniform sea of background molecules, and R_p, which represents the penetration distance of a background molecule into the source gas. In terms of the variables used in this paper, we find that

$$R_\infty = O(\alpha/P), \quad R_p = O(1/\alpha).$$

Further it is assumed that $R_p \ll R_\infty$.

They consider two regions where

$$\tilde{R} = \frac{r/r_*}{R_p} = O(1), \quad R^* = \frac{r/r_*}{R_\infty} = O(1).$$

This first region is seen to be the region of the shock layer, where $Y = O(1)$, and the second a region where $Y = O(P/\alpha^2) = O(W_0^{-1})$. The latter region is small compared with the former since $R_p \ll R_\infty$ implies $P/\alpha^2 \ll 1$. It will be seen therefore that a region whose thickness is of order $(W_0^{-\frac{3}{2}})$ is not required as in this paper. This is, of course, most probably due to the special nature of the breakdown for the Navier–Stokes equations. However, since the Boltzmann equation gives rise to a much more complex situation, this point is worthy of further study.

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